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Interceptor Line-of-Sight Rate Steering: Necessary Conditions for a Direct Hit

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The zero-effort miss of an interceptor is a vector that measures the closest pass to the target that will occur on its current trajectory if no corrective control is applied. The position divert is the change in the zero-effort miss that occurs as a result of applying control. A necessary condition for a zero miss is that the magnitude of the interceptor zero-effort miss be smaller than that of the maximum position divert at all values of interceptor–target range. Using this condition, a criterion for a hit is established: The set of probable zero-effort misses must always be a subset of the set of position diverts. Because it is impossible to meet this condition for arbitrarily small interceptor–target range, there must be some critical range at which the guidance fails. The work is limited to predictive homing guidance using reaction control and shows how the guidance failure mechanism influences the interceptor miss. A simple approximate expression for miss distance is obtained, in terms of homing seeker noise, maximum divert acceleration, and uncertainty about target acceleration.

I. Introduction

THERE are five main processes that limit the smallest miss that can be guaranteed by a homing interceptor against a target such as a ballistic missile. They are 1) the initial uncertainty about the interceptor–target state; 2) the rate at which the target can unpredictably change its state, i.e., maneuver; 3) the rate at which state uncertainty grows due to the autonomous dynamics of the interceptor–target system; 4) the rate at which information can be gathered about the system state by the interceptor guidance seeker measurements, determined by seeker measurement accuracy; and 5) the rate at which the interceptor can alter its heading, i.e., divert, in response to guidance controls.

A simple (but admittedly intuitive) theory is developed that explains how some of these processes interact in a homing guidance system employing pure reaction sidethrust control. It is assumed that the sidethrusters operate instantaneously on command. A heuristic algebraic expression is derived that suggests the likely miss distance uncertainty, or miss grouping, for the case where the miss process is dominated by the interaction of state variable estimation uncertainty with hard limiting of divert thrust. The miss uncertainty is a function of closing velocity, homing seeker angular resolution, maximum interceptor divert acceleration, and uncertainty about target acceleration.

The principal aim of the paper is to find a concise way of explaining the effect of divert thrust hard limiting, and its interaction with uncertainty in the system state, on the miss distance of a reaction sidethrust controlled homing interceptor. Monte Carlo methods (repeated simulation trials with ensemble averaging) are inexpensive in these days of fast computers, but they yield only numerical descriptions of miss distance distributions: they do not yield the underlying causes of miss that are sought; i.e., they do not yield a clear explanation of the miss process. Computationally efficient alternatives to Monte Carlo methods, using statistical linearization,[1] have been employed in the past. Gelb and Warren[2] used a covariance analysis describing function technique (CADET) to get direct numerical statistical performance projections. Zarchan[3] used the CADET method to compute describing function gains, which were then stored and used as time-varying coefficients in a numerical adjoint analysis. The technique was coined the statistical linearization adjoint method (SLAM).

CADET and SLAM give clues to the explanation sought, but their output is numerical. The work here does not use statistical linearization, but rather a reachable set idea[4]: To guarantee a hit, the set of probable interceptor zero-effort misses must be a subset of the set of interceptor position diverts.

II. Fundamental Ideas

A. Target Estimated Set

Suppose that an estimate of the full state of the target (its position, velocity, attitude, attitude rate, etc.) is available, supplied by a state estimator that is driven by a seeker observing the target. The state estimate is characterised by a probability density function (PDF) in the target state space, which is conditional on all past seeker measurements. For the purposes of explanation here, and without much loss of generality, consider this PDF to have a simple property: All of the possible states of the target that are admitted by the past seeker measurements are equally likely. This assumption is made solely to shorten the description of the ideas. In practice, the PDF is likely to be Gaussian or near-Gaussian. Certainly, if a Kalman filter is employed for state estimation, the true system state PDF will be approximated by a Gaussian distribution. The bounds of the uniform distribution used here might be replaced by a 3σ specification in the Gaussian case.

The set of possible target states admitted by the uniform PDF shall be called the estimate set E. The true state of the target is a member of E, i.e., it lies within E to probability 1. The bounds of E are determined by the measurements on the target state, made by the interceptor homing seeker. Ideally, E should be small.

Setting aside unpredictable atmospheric variations and input-induced maneuver, the differential equation for the target state is autonomous. Therefore, each state in E is the initial state of a possible target trajectory.

Of key importance is whether these possible target trajectories are interceptable. The interceptor, with zero control effort applied, i.e., on a coasting trajectory, makes a closest approach with each of the possible target trajectories. The vector of closest approach shall be called a zero-effort miss (ZEM) vector.

At ranges much greater than the magnitude of the ZEM vector, the ZEM vector is almost normal to the interceptor–target line of sight (LOS). Thus, the component of the ZEM vector along the LOS is almost zero. The components of the ZEM vector normal to the LOS may be positive or negative, inducing LOS rate components of corresponding sign. For simplicity of description, consider only one plane of the guidance and one corresponding normal component of the ZEM vector. Denote the component simply as the ZEM, a scalar that can take either sign. The probability of being able to hit the target cannot be 1 unless every one of the possible ZEMs is nullable

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by an admissible interceptor control sequence because one of the possible ZEMs is the true ZEM.

B. Interceptor Reachable Set

Given the current interceptor state and knowledge of its response to controls, there must be some computable set of initial target states that correspond to a set of interceptable, or reachable, trajectories. Call this set R, the reachable set. By definition, the interceptor ZEMs against the set R are all nullable by admissible controls. The bounds of R are constrained by the maximum available interceptor divert acceleration. Ideally, R should be large.

C. Intersection Set and Hit Probability

The target trajectories corresponding to the states of E that lie within R can be intercepted. Those trajectories that lie outside R cannot be intercepted. If the true target state, in fact, lies outside R, then an intercept is not possible.

Thus, the target trajectories that are interceptable have initial states in some intersection region I of E and R. Because the true target state lies within E to probability 1, and E has a uniform PDF, the probability of being able to hit the target cannot be greater than the ratio of the number of states in I to that in E. For a hit-to-kill interceptor, the performance objective is a hit probability of 1, i.e., that the miss is always zero. A necessary condition for this is the estimate set E must always be a subset of the reachable set R, whatever the range between the interceptor and the target. It turns out that this objective is impossible to meet and that the probability of a zero miss can never actually be 1.

III. ZEM and Interceptor Divert

A. ZEM and Uncertainty

The preceding development suggests how uncertainty in the target state might interact with the limit on interceptor acceleration divert to cause a possible loss of hit probability with respect to the ideal value of 1. The hit probability must fall below 1 as soon as the set E ceases to be a subset of R.

Define the sample miss in the guidance plane to be a normal component of the vector of closest pass between the interceptor and the target. The time of intercept is taken to be the time of closest pass. If the interceptor control is fixed at zero throughout the sample trajectory, the trajectory is a zero-effort trajectory and the sample miss is the sample ZEM.

For each of the states in the estimated target set E, a sample zero-effort relative trajectory may be constructed. A set of sample trajectories results, and each has an associated sample ZEM. A situation will be considered in which the E set states have common range values. Effectively, this means that the interceptor–target range is known.

B. Maximum Divert Limitation

If the interceptor divert acceleration is limited, due to sidethrust limitation in the divert thrusters, there will be a limit to the position divert and, hence, a limit to the ZEM that can be nulled by applying sidethrust.

For example, if the time to go to closest pass is t_{go}, the maximum divert acceleration normal to the interceptor–target LOS is a_{max}, and the sidethrusters act instantaneously on command, then the maximum position divert following a step demand at time to go t_{go} is just

$$s = 0.5\, a_{max} t_{go}^2 \tag{1}$$

This is the maximum ZEM that can be nulled at time to go t_{go}, and because

$$t_{go} \simeq r/V \tag{2}$$

where r is the interceptor–target range and V is the closing speed, the maximum tolerable ZEM is just

$$Z_{max} \simeq 0.5\, a_{max} r^2/V^2 \tag{3}$$

Superimposing the position divert limit envelope and the set of sample ZEMs (the ZEM distribution) on the same diagram yields

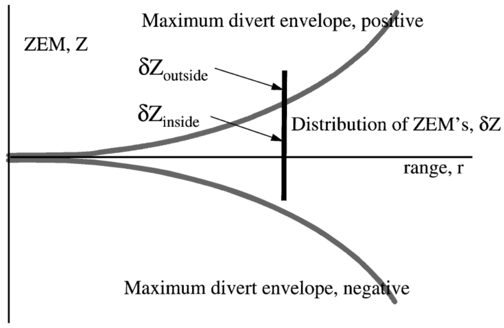

Fig. 1 Distribution of ZEMs and position divert.

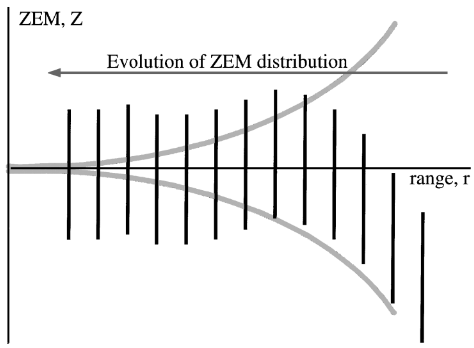

Fig. 2 Effect of interceptor control, without seeker measurements.

Fig. 1. As shown in Fig. 1, the ZEM distribution has a region δZ_{inside}, which is inside the maximum position divert envelope; if the true ZEM lies here a hit is possible. But it also has a region $\delta Z_{outside}$, which is outside the divert envelope. If the true ZEM lies in $\delta Z_{outside}$ the target cannot be hit. The hit probability cannot be greater than

$$p = \frac{\delta Z_{inside}}{(\delta Z_{inside} + \delta Z_{outside})} \tag{4}$$

IV. Necessary Conditions for a Guaranteed Zero Miss

A. Necessary Condition

It is clear from Fig. 1 that a necessary condition at some range r to guarantee a zero miss is that the whole distribution of sample ZEMs at the range r must lie inside the maximum position divert envelope, for if any part of the ZEM distribution lies outside this envelope, the target may be unreachable. A sufficient condition for a zero miss is that the ZEM distribution must lie inside the maximum position divert envelope for all ranges up to intercept.

B. Dynamics of the ZEM Distribution

The dynamics of the ZEM distribution are determined by three factors: 1) target maneuver, 2) control applied to the interceptor, and 3) measurements provided to the state estimator.

In the absence of input-induced target maneuver, the dynamics of the ZEM are zero. The ZEM distribution is, thus, constant if both interceptor control and seeker measurements are zero (control determines the position of the mean of the ZEM distribution, whereas seeker measurements determine its variance).

C. Effect of Interceptor Control Without Seeker Measurements

Assuming use of the mean of the ZEM distribution to construct a control sequence for the interceptor, the effect of the control is to drive the whole ZEM distribution so that its mean moves toward zero, as shown by the sequence of Fig. 2. Assuming seeker measurements are absent, nothing is learned about the ZEM, the ZEM uncertainty is constant, and some part of the ZEM distribution eventually falls outside the interceptor position divert envelope. The probability of a hit at any given range cannot be greater than the fraction of the ZEM distribution that lies inside the position divert envelope.

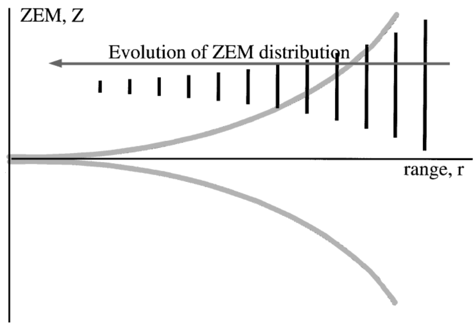

Fig. 3 Effect of seeker measurements, without interceptor control.

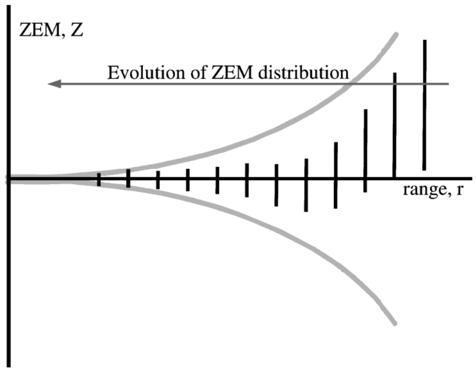

Fig. 4 Effect of control and measurements.

D. Effect of Seeker Measurements Without Interceptor Control

The effect of seeker measurements is to reduce the uncertainty of the state estimates, and so the distribution of ZEMs gets narrower, as suggested by Fig. 3. Assuming control is absent, the ZEM distribution as a whole is not driven toward zero, and it eventually falls entirely outside the interceptor position divert envelope. Once this happens the probability of a hit is zero.

E. Combined Effect of Control and Measurements

The combined effect of control and measurements is shown in Fig. 4. In the example shown, the progressive reduction in uncertainty about the ZEM due to measurement information and the movement of the center of the ZEM distribution toward zero due to the control, both tend to hold the ZEM distribution within the interceptor position divert envelope.

F. Necessary and Sufficient Conditions for a Guaranteed Zero Miss

Provided the ZEM uncertainty contraction process and the ZEM center nulling process both continue until the range is zero, so that the whole ZEM distribution always lies inside the divert boundary, a zero miss will be guaranteed.

The following conditions must be met: 1) The initial ZEM distribution must lie entirely within the interceptor position divert envelope. 2) The ZEM uncertainty must reduce with r^2 as range r falls, otherwise some part of the ZEM distribution may eventually fall outside the divert envelope (which reduces in proportion to r^2). It will be shown later that this requirement is impossible to meet below some critical value of r. 3) The ZEM uncertainty must eventually be nulled. This requires an infinite amount of measurement information, which is impossible. 4) The control process must ensure that the mean ZEM is reduced in proportion to r^2, otherwise some part of the ZEM distribution may eventually fall outside the divert envelope. This requirement may be difficult or impossible to meet if there are control lags and hardware nonlinearities in the interceptor dynamic system.

V. Necessary Conditions for a Small Miss

A. Definition of a Small Miss

Provided the miss with respect to some reference point in the target can be guaranteed to be small compared with the physical size of the target, a hit can be claimed. A small miss is the same as a hit. Therefore, the difficulty in guaranteeing a small miss, where small means less than the physical size of the target, is to be considered next.

B. Guidance Critical Range

Provided the distribution of sample ZEMs lies inside the interceptor position divert envelope, an intercept is possible. But once some part of it falls outside the envelope, the probability of a hit falls below 1 [Eq. (4)]. If most of the ZEM distribution lies outside the envelope, the probability of a hit is small. Very little advantage exists in applying an interceptor control once this situation is reached; the ZEM uncertainty is larger than the available divert, so that most controls computed from the ZEM set will not have significant effect in improving the miss. The guidance is almost ineffective. The guidance system might as well apply zero control. The sample misses then equate to the sample ZEMs.

At this critical range, the distribution of sample misses becomes almost the same as the distribution of ZEMs, and because the actual ZEM is a member of the set of sample ZEMs, the actual miss may be approximated by a sample from the ZEM distribution. The guidance critical range, and an expression to compute it, will be considered in Sec. VI.

Suppose the target size is of the order of 1 m. Then define (arbitrarily) a small miss as 0.1 m, so that the tolerable uncertainty in the ZEM at the guidance critical range is 0.1 m. The states that determine the ZEM must be estimated to sufficient accuracy to guarantee that this uncertainty is not exceeded.

C. Description of the ZEM

Suppose the target velocity is constant and the interceptor flies at constant velocity under zero control effort. Then the ZEM at range r is

$$z = \omega_s r^2 \big/ V \tag{5}$$

where ω_s is the LOS rate and V is the speed on the relative trajectory. To compute an estimate of the ZEM, estimates of ω_s, r, and V are needed.

In tactical missile guidance, it is common practice not to estimate the ZEM explicitly. The problem is to construct a lateral acceleration control for the interceptor that will null the ZEM,[5–7] without actually calculating the ZEM. In the time to go t_{go}, the ZEM must be nulled with an accelerative control. One possible proportional control might be

$$\begin{aligned} a_{\text{control}} &= \frac{Kz}{t_{go}^2} \\ &= \frac{K\omega_s r^2}{V t_{go}^2} \end{aligned} \tag{6}$$

where K is some gain constant. Because

$$t_{go} \simeq r/V \tag{7}$$

the control becomes

$$a_{\text{control}} = K V \omega_s \tag{8}$$

Equation (8) is the well-known proportional navigation (PN) guidance law. K is the navigation factor, which is usually set at about 3 or 4. To implement the law, estimates of ω_s and V are needed. An estimate of range r is not needed.

If the target is accelerating, or the interceptor zero-effort trajectory is not constant velocity, or both, the ZEM may be written approximately as

$$z \simeq \omega_s r^2 \big/ V + f(r, \boldsymbol{x}) \tag{9}$$

where f is the contribution to the ZEM due to the accelerative motion and $\boldsymbol{x}(r)$ is a vector of states, which is sufficient to describe the accelerative motion, given the dynamics of the motion.

As with the PN example, a proportional control may be constructed by dividing by t_{go} squared, which will be influenced by relative acceleration. Depending on the dynamics of the acceleration, knowledge of the range r to compute the control may or may not be necessary. To construct the control, the dynamics of the acceleration must be known, and ω_s and $\boldsymbol{x}$ must be estimated. In practice, the dynamics may not be precisely known, but enough information about them may be available to represent them reasonably confidently in the critical end-game period by simple functions or series approximations.

Other types of control may be constructed, for example, if the interceptor uses a divert motor that produces a fixed thrust for a controllable duration. We will not go into such schemes here.

For the purposes of this work, assume that the ZEM can be represented by Eq. (9). If Eq. (9) is not valid the computed ZEM distribution will be biased, and this may induce a miss, depending on whether the bias is range dependent. If the uncertainty in the representation is known then it may be incorporated in the ZEM distribution (Fig. 1). The effect of the additional uncertainty is to increase the guidance critical range.

D. Tolerable Uncertainty in the LOS Rate

If the LOS rate is uncertain by $\delta\omega_s$, then by Eq. (9), the corresponding uncertainty in the ZEM is

$$\delta z_\omega = \frac{\delta\omega_s r^2}{V} \tag{10}$$

To guarantee a miss smaller than the interceptor maximum position divert at range r, the whole ZEM distribution and, hence, δz_ω must be contained within the interceptor position divert envelope. Were the uncertainty in LOS rate to be constant, i.e., independent of range, the ZEM uncertainty would vary as the square of range, the same as the divert. But LOS rate uncertainty is not constant; it tends to rise in the end game, as we shall see. Therefore, the ZEM uncertainty does not vary with range in the same way as the divert. We shall see how it does vary later.

Requirements on acceptable LOS rate uncertainty to meet the guarantee are

$$\delta z_\omega \le \frac{a_{\max} t_{go}^2}{2} \tag{11a}$$

that is,

$$\delta z_\omega \le \frac{a_{\max} r^2}{2V^2} \tag{11b}$$

that is,

$$\frac{\delta\omega_s r^2}{V} \le \frac{a_{\max} r^2}{2V^2} \tag{11c}$$

that is,

$$\delta\omega_s \le \frac{a_{\max}}{2V} \tag{11d}$$

The whole LOS rate distribution and, hence, the LOS rate uncertainty must be contained within an envelope determined by the interceptor divert acceleration divided by the relative speed. This is a necessary condition for a satisfactory engagement, but not a sufficient one because there are other contributions to ZEM uncertainty.

In a conventional tactical missile application, $a_{\max}$ and V might be expected to be 20 g and 1000 m/s, respectively, requiring LOS rate uncertainty to be less than 0.2 rad/s. But in ballistic missile defense $a_{\max}$ might be limited to 3 g, and V is typically 3000 m/s. The LOS rate must be estimated to better than 0.01 rad/s.

The design of a state estimator/terminal homing seeker to meet this requirement may be difficult, particularly in the endo-atmospheric interception case, where the target motion may be non-Keplerian and only partially predictable. The specifications on the seeker and the on-board inertial instruments, which must supply the information to enable the state estimator to converge its LOS rate estimate, will be very exacting by comparison with the tactical missile case. It is likely that the tolerance to seeker measurement errors caused by window and flowfield aberrations will be about an order of magnitude more stringent.

A heuristic expression that predicts the effect on the interceptor miss distance due to seeker measurement error, state estimator uncertainty about target acceleration, and interceptor divert acceleration limiting will be established next. The method of deriving the expression is unconventional. It is admitted that the expression gives only an approximate guide to the likely miss.

VI. Information, Uncertainty, and Miss Distance

A. General

The arguments that will be used to establish interceptor miss are semi-intuitive. However, they do lead to results that tend to be confirmed by computer experiments. Consider the terminal homing phase. Information to estimate the ZEM, or the state variables that constitute it, must come from the terminal homing seeker. The information is used to reduce the uncertainty in the ZEM. Some sample from the ZEM distribution, possibly the mean value, must be used to generate a control for the interceptor, so that the ZEM distribution is guided collectively toward zero.

Estimation of the ZEM depends on estimation of the LOS rate [Eq. (5)]. The dynamics of the ZEM are connected with those of the LOS rate. The center value of the LOS rate distribution is determined by the control applied to the interceptor. The uncertainty of the distribution is determined by the measurements from the interceptor seeker.

B. Dynamics of LOS Rate

Consider one plane of the guidance. The dynamics of the LOS rate ω_s are given by

$$\frac{d\omega_s}{dt} = -2\frac{\dot{r}}{r}\omega_s + \frac{1}{r}(a_t - a_m) \tag{12}$$

where r is the interceptor–target range, $\dot{r}$ is the range rate, and a_t and a_m are the target and interceptor accelerations, normal to the LOS.

In simple PN guidance, ω_s is driven to zero by the control on a_m, by setting

$$a_m = -K\dot{r}\omega_s \tag{13}$$

where

$$\dot{\omega}_s = (K-2)(\dot{r}/r)\,\omega_s + (a_t/r) \tag{14}$$

Because $\dot{r}$ is negative, the LOS rate will be driven toward zero provided the target acceleration is not too large and the navigation factor K is greater than 2.

However, this result is not generally applicable when there is a limit on the interceptor available lateral acceleration. Earlier, it has been shown that for a hit probability of 1 the whole LOS rate distribution must lie inside an envelope $\pm a_{\max}/2V$ and, consequently, so must any sample ω_s drawn from that distribution.

C. Dynamics of LOS Rate Uncertainty

From Eq. (12), a differential equation for LOS rate uncertainty can be developed. Suppose that a state estimator such as a Kalman filter is used to estimate (among other states) LOS rate and target acceleration. In an interval between seeker discrete measurements, the LOS rate uncertainty will evolve as

$$\frac{d\delta\omega_s}{dt} = -2\frac{\dot{r}}{r}\,\delta\omega_s + \frac{1}{r}\,\delta a \tag{15}$$

where δa is the uncertainty in target acceleration. The range and range rate may also be uncertain in practice, particularly if passive homing is used, but the effect of these uncertainties will not be considered here.

The rate of change of LOS rate uncertainty is inversely proportional to range r. Uncertainty will be reduced impulsively at each discrete seeker measurement. It is desired to constrain it to within the limits $\pm a_{\max}/2V$ until as late in the interception as possible, and this puts an information rate requirement on the seeker. Consider a range at which the rate of uncertainty growth defined by Eq. (15) exceeds the seeker information rate. As the range falls, the uncertainty grows rapidly, and quickly exceeds the desired $a_{\max}/2V$ limit.

The ZEM uncertainty grows correspondingly. As it begins to exceed the maximum position divert, the guidance starts to become ineffective, and a miss which may be approximated by some sample from the ZEM distribution becomes inevitable. The later in the engagement this process occurs, the smaller will be the variance of the miss distribution.

D. Effect of Seeker Measurements in Reducing LOS Rate Uncertainty

The problem of getting a simple expression for the guidance critical range will be approached by establishing a characteristic time interval T_{detect} for the seeker to measure a rate to an accuracy of the order of $a_{\max}/2V$.

If the increase in LOS rate uncertainty due to the dynamics in the interval T_{detect} is much less than $a_{\max}/2V$, then the information from the seeker in the interval will cause a reduction in LOS rate uncertainty.

If the increase in LOS rate uncertainty due to the dynamics in the interval T_{detect} is of the order of $a_{\max}/2V$, then the information from the seeker in the interval will just balance the increase in uncertainty due to the dynamics and the LOS rate uncertainty will be approximately constant.

If the increase in LOS rate uncertainty due to the dynamics in the interval T_{detect} is greater than $a_{\max}/2V$, then the information from the seeker in the interval will be insufficient to offset the increase in uncertainty due to the dynamics and the LOS rate uncertainty will increase. In this case, the seeker cannot supply information fast enough to prevent the LOS rate uncertainty from increasing: The uncertainty will, therefore, grow uncontrollably in this and succeeding intervals, and will quickly exceed the $\pm a_{\max}/2V$ limits.

For continuous measurements with Gaussian noise, the uncertainty evolution is prescribed by a matrix Riccati differential equation, which may be factorised into two linear matrix differential equations and solved analytically under certain restrictive conditions. For discrete measurements, a rigorous analytic solution seems to be more difficult.

The seeker measures the look rotation from some reference to the LOS. Suppose the reference is determined by an expected LOS rate. Then the change in the rotation in some interval T_{detect} for a sample LOS rate drawn from the distribution of LOS rates is

$$\delta\theta_{\text{look}} = \delta\omega_s \cdot T_{\text{detect}} \tag{16}$$

The seeker will be able to detect this change only if it is greater than some threshold θ_{seeker} determined by such factors as the seeker noise level, the pixel size, the size of the target image, and so on. Thus, for the change to be detectable, the interval T_{detect}, which will be called the seeker detection time, must be

$$T_{\text{detect}} = \frac{\theta_{\text{seeker}}}{\delta\omega_s} \tag{17}$$

The time required for the seeker to deliver LOS rate information to accuracy $a_{\max}/2V$ is, thus,

$$T_{\text{detect}} = \frac{\theta_{\text{seeker}}}{a_{\max}/2V} = \frac{2V\theta_{\text{seeker}}}{a_{\max}} \tag{18}$$

In this time, the dynamics growth in LOS rate uncertainty, from Eq. (15), is

$$\Delta(\delta\omega_s) \simeq \frac{\mathrm{d}\delta\omega_s}{\mathrm{d}t} T_{\text{detect}}$$

$$= \left(-2\frac{\dot{r}}{r}\delta\omega_s + \frac{1}{r}\,\delta a\right) T_{\text{detect}} \tag{19}$$

When the range is much larger than the range rate, the growth will be small. The sequence of seeker measurements should reduce the LOS rate uncertainty to $a_{\max}/2V$ or less. If the LOS rate uncertainty cannot be reduced to $a_{\max}/2V$ or less at any point in the interception, then the miss grouping will be larger than the value we shall compute here, because some part of the ZEM distribution will consequently always be outside the maximum divert envelope.

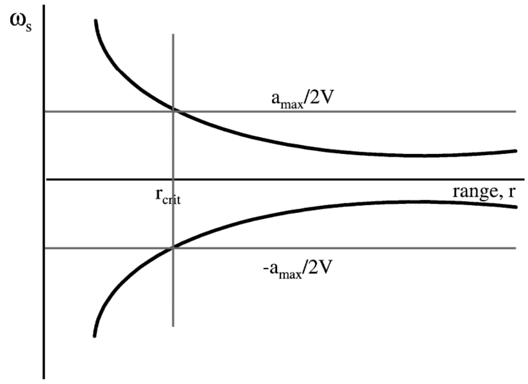

Fig. 5 Evolution of LOS rate uncertainty.

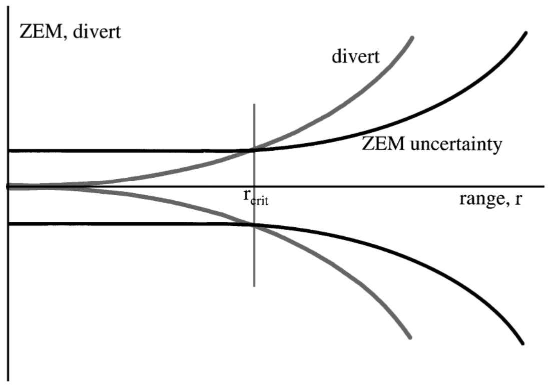

Fig. 6 Evolution of ZEM uncertainty.

As r approaches the guidance critical range, the growth in LOS rate uncertainty will become equal to the reduction due to the seeker measurements, and as r falls below the critical range, the growth will exceed the seeker reduction. The LOS rate uncertainty will then begin to grow rapidly. Therefore, the guidance will begin to fail, and the miss distribution will be similar to the ZEM distribution at the critical range. Figure 5 is intended to illustrate the phenomenon. The region of LOS rate uncertainty lies between the curved lines. In this example, the central value of the uncertainty distribution is zero, and at long range the rate of uncertainty growth due to the dynamics is less than the rate at which the seeker supplies information. The LOS rate uncertainty, therefore, falls. Later, it becomes near constant, implying that the seeker information rate just balances the dynamics growth rate. At the critical range r_{crit}, the dynamics growth rate exceeds the seeker information rate, and the uncertainty becomes uncontrollable; it quickly exceeds the desired $a_{\max}/2V$ limit. Because the ZEM uncertainty is linked to the LOS rate uncertainty via Eq. (10), its behavior must be similar to that shown in Fig. 6.

Figure 6 suggests that the ZEM uncertainty is constant for ranges less than r_{crit}, which is consistent with the following argument. By Eq. (15) the rate of growth of LOS rate uncertainty after the critical range r_{crit}, when the seeker measurements cease to have any significant influence, is approximately equal to

$$\frac{\mathrm{d}\delta\omega_s}{\mathrm{d}t} = -2\frac{\dot{r}}{r}\,\delta\omega_s \tag{20}$$

If the coefficient $\dot{r}/r$ of $\delta\omega_s$ were constant, Eq. (20) would have an exponential solution. But the coefficient changes rapidly in the end game. A solution of Eq. (20) is found to be

$$\delta\omega_s = \delta\omega_s(r_{\text{crit}})\left(\frac{r_{\text{crit}}}{r}\right)^2 \tag{21}$$

Substituting this result into Eq. (10), the part of the ZEM uncertainty due to LOS rate uncertainty is obtained as

$$\delta z_\omega = \frac{\delta\omega_s(r_{\text{crit}})\,r_{\text{crit}}^2}{V} \tag{22}$$

which is independent of range r.

In truth, the seeker will supply a small amount of information after $r_{\rm crit}$, and the ZEM uncertainty may go on falling. But because the uncertainty is larger than the maximum position divert, little can be done about the miss. Whether or not the ZEM uncertainty is constant, $r_{\rm crit}$ is the range at which the ZEM uncertainty envelope crosses the maximum position divert envelope.

E. Expressions for Guidance Critical Range and Miss Uncertainty

Rapid growth of LOS rate uncertainty with respect to the limits $\pm a_{\max}/2V$ will begin when, in the absence of seeker measurements, the growth of uncertainty in a $T_{\rm detect}$ interval becomes greater than $a_{\max}/2V$. Then, as discussed at the start of Sec. VI.D, the information provided by the seeker at a measurement is insufficient to negate the increase in uncertainty. By Eqs. (11) and (19), rapid growth will begin when

$$\delta\omega_s \simeq a_{\max}/2V \tag{23}$$

and

$$[-2(\dot r/r)\delta\omega_s + (1/r)\delta a]T_{\rm detect} \simeq \delta\omega_s \tag{24}$$

Let $r_{\rm crit}$ be the range at which this happens. Substituting for the variables,

$$r_{\rm crit} \simeq \frac{4V^2\theta_{\rm seeker}}{a_{\max}}\left(1+\frac{\delta a}{a_{\max}}\right) \tag{25}$$

Because the ZEM uncertainty is approximately

$$\delta z \simeq \frac{\delta\omega_s r_{\rm crit}^2}{V} + \frac{1}{2}\delta a\frac{r_{\rm crit}^2}{V^2} \tag{26}$$

and the miss uncertainty is approximately equal to the ZEM uncertainty beyond this point, it is concluded that the miss uncertainty will not be much less than

$$\delta m \simeq \frac{8V^2\theta_{\rm seeker}^2}{a_{\max}}\left(1+\frac{\delta a}{a_{\max}}\right)^3 \tag{27}$$

This result suggests that the set of possible misses lies in some range that is not smaller than δm, the center value of which should be near zero (because of the effect of the interceptor control in centering the distribution). The actual miss is a member of the set and could be as large as δm; it cannot be guaranteed to be less than δm. Therefore, it must be ensured that δm is less than the criterion for a hit, for example, 0.1 m.

F. Examples

1. Example 1

Suppose the closing speed is $V = 3000$ m/s, the interceptor maximum divert acceleration is $a_{\max} = 3\ g$, the seeker angular resolution is $\theta_{\rm seeker} = 0.3$ mrad, and the uncertainty in the target acceleration normal to the LOS is 1 g. Then the uncertainty in the miss in meters is

$$\delta m \simeq 0.5 \tag{28}$$

This uncertainty is larger than might have been hoped for. To guarantee a miss of 0.1 m or less requires an improvement to the seeker resolution of about $\sqrt{5}$, i.e., to about 0.13 mrad, which will be difficult and expensive. Alternatively, the interceptor divert acceleration $a_{\max}$ could be increased to 7 g.

2. Example 2

With δa zero, implying perfect knowledge of target acceleration,

$$\delta m \simeq \frac{8V^2\theta^2}{a_{\max}} \tag{29}$$

Thus, with $V = 3000$ m/s, $\theta_{\rm seeker} = 0.3$ mrad, it is not possible to achieve the 0.1-m miss uncertainty unless the maximum divert acceleration $a_{\max}$ is at least 6.6 g.

3. Example 3

In the terminal stage of homing, the target image may cover many pixels on the focal plane array of an infrared seeker. What is the effect of this image size? In the absence of image processing designed to select a particular feature in the target image to single pixel accuracy, the effective seeker resolution is not one pixel, but rather

$$\theta_{\rm seeker} \simeq d_t/2r \tag{30}$$

where d_t is the target size. The effective seeker detection time, Eq. (18), becomes

$$T_{\rm detect} \simeq \frac{V d_t}{a_{\max} r} \tag{31}$$

For simplicity of calculation, assume that the uncertainty in the target acceleration is zero, and rework the calculations for critical range. This yields

$$r_{\rm crit} \simeq V\sqrt{2d_t/a_{\max}} \tag{32}$$

The miss distance uncertainty becomes

$$\delta m = \frac{\delta\omega_s r_{\rm crit}^2}{V} = \frac{a_{\max}}{2V}\frac{r_{\rm crit}^2}{V} \simeq d_t \tag{33}$$

i.e., the miss uncertainty is of the order of the target size.

VII. Conclusions

A simple (and predominantly intuitive) theory has been developed that explains how limitations on guidance seeker information rate and interceptor divert influence the miss distance of a homing interceptor employing reaction sidethrust control. The implications of the work are as follows:

1) To guarantee a zero miss (with respect to some designated point in the target), the distribution of sample ZEMs computed from the interceptor–target state PDF must lie inside the maximum position divert envelope, at all values of interceptor–target range.

2) No matter what guidance law is used, there exists a critical range at which the distribution of sample ZEMs ceases to lie entirely inside the divert envelope, due to seeker information rate limitations (determined by noise, pixelization, sampling rate, etc.). The subset of ZEMs that lies outside the position divert envelope contains sample ZEMs that are nonnullable. Because the actual ZEM may be a member of this nonnullable set, it is impossible to provide a guarantee of zero miss.

3) The grouping of misses approximates the ZEM distribution at the critical range, because the gquidance tends to become ineffective at ranges less than the critical range.

4) It may be possible to design the interceptor guidance so as to ensure that the miss grouping is smaller than the target size. Thus, though it may not be feasible to guarantee a zero miss, it may be possible to guarantee a hit somewhere on an extended target.

The work is significant because it indicates how the miss grouping is related to closing velocity, seeker measurement accuracy, maximum interceptor maneuver acceleration, and state estimator uncertainty about target acceleration. The results obtained, though admittedly approximate, may be used to indicate the seeker measurement accuracy and interceptor divert acceleration specifications that are necessary to guarantee that the miss distance grouping does not exceed some desired value.

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